1/28/20

"More Parameters" (continued)

9) Excess Loss Random Variable (d-deductible)

$X - d \mid X > d$ = r.v. amount paid by the insurance company, given that an insurance payment is made

This r.v. is called <u>truncated</u> ($X > d$ part) and <u>shifted</u> ($X - d$ part)

General Statistics Facts:

Given the pdf $f_X(x)$ for the r.v. X, then

$$f_{X|X>d}(x) = \begin{cases} 0 & \text{if } x \leq d \\ \dfrac{f_X(x)}{S_X(d)} & \text{if } x > d \end{cases}$$

$$E[g(X) \mid X > d] = \int_0^\infty g(x) \cdot f_{X|X>d}(x)\, dx$$

$$= \int_d^\infty g(x) \cdot \frac{f_X(x)}{S_X(d)}\, dx$$

Let $g(X) = X - d$.

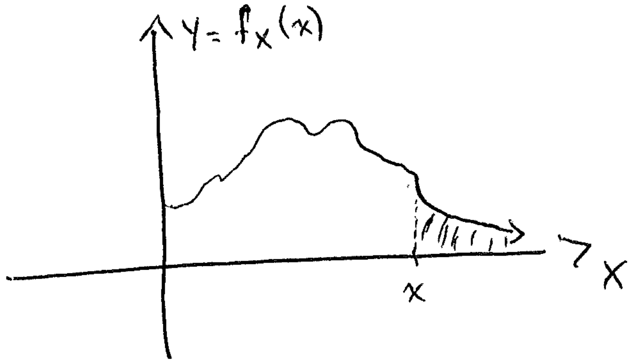
$$E[X - d \mid X > d] = \text{mean excess loss} = e_X(d)$$

$$= \int_d^\infty (x-d) \cdot \frac{f_X(x)}{S_X(d)}\, dx$$

Note: $e_X(d) \overset{IBP}{=} \int_0^\infty \frac{S_X(x+d)}{S_X(d)}\, dx \quad \left(= \int_d^\infty \frac{S_X(x)}{S_X(d)}\, dx \right)$

= expected insurance payment, given that a payment is made

Tail Behavior



"right tail"

$S_X(x) = \Pr(X > x)$

Comparing two different distributions' tails

Notation:

$+$ denotes the distribution with the heavier tail

$-$ ——————————————— lighter tail

Then: 1) $\dfrac{f_-(x)}{f_+(x)} \xrightarrow{x \to \infty} 0$

2) $\dfrac{S_-(x)}{S_+(x)} \xrightarrow{x \to \infty} 0$

3) $\dfrac{h_-(x)}{h_+(x)} \xrightarrow{x \to \infty} \infty$

Q: Why?

A: $\dfrac{h_-(x)}{h_+(x)} = \dfrac{f_-(x)}{f_+(x)} \cdot \dfrac{S_+(x)}{S_-(x)} \left(\overset{x \to \infty}{=\!=} 0 \cdot \infty \right)$

dominates

Other Facts:

1) The k^{th} moment ($E[X^k]$) exists for larger k for $-$ than for $+$

2) "X has a light tail" when

   1) $h_X(x)$ is an increasing function of x

   or

   2) $e_X(d)$ is a decreasing function of d

Examples:

1) $X \sim Exp(\Theta)$

Then $h_X(x) = \frac{f_X(x)}{S_X(x)} = \frac{\frac{1}{\Theta} e^{-x/\Theta}}{e^{-x/\Theta}} = \frac{1}{\Theta}$ (constant w.r.t x)

$\therefore$ the exponential distribution is neither light tail or heavy tail.

The exponential distribution is the standard that other distributions are compared to in order to determine whether they are light or heavy tailed.

Note: $X \sim Exp(\Theta) \Rightarrow (X-d \mid X>d) \sim Exp(\Theta)$

$\therefore e_X(d) = E[X-d \mid X>d] = \Theta$ (constant w.r.t. d)

2) $X \sim$ 2-Pareto (α, θ)

Then $h_X(x) = \dfrac{f_X(x)}{S_X(x)} = \dfrac{\alpha \cdot \theta^\alpha}{(x+\theta)^{\alpha+1}} \cdot \dfrac{(x+\theta)^\alpha}{\theta^\alpha} = \dfrac{\alpha}{x+\theta}$

is a decreasing function of x.

$\therefore$ the 2-Pareto distribution is heavy tailed

Remark: Consider $\underbrace{X-d \mid X>d}_{Y}$

Note: $S_Y(y) = Pr(Y>y) = Pr((X-d \mid X>d)>y)$

$= Pr(X-d>y \mid X>d)$

$= Pr(X>y+d \mid X>d)$

$= \dfrac{S_X(y+d)}{S_X(d)} = \dfrac{\left(\frac{\theta}{y+d+\theta}\right)^\alpha}{\left(\frac{\theta}{d+\theta}\right)^\alpha}$

$= \left(\dfrac{d+\theta}{y+d+\theta}\right)^\alpha$

$\Longrightarrow (X-d \mid X>d) \sim$ 2-Pareto $(\alpha, d+\theta)$